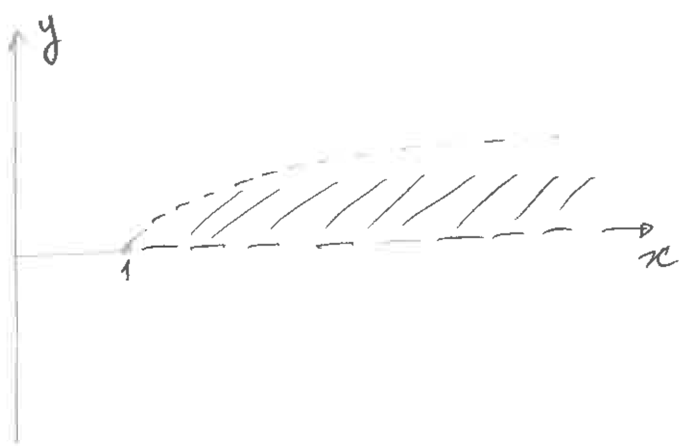
Mathematical Economics - Resit Exam -

- 08/01/2026 - Version A

Part I.

a) $D_f = \{(x,y) \in \mathbb{R}^2 : x - y^2 - 1 > 0 \ \wedge \ y > 0\}$

$y^2 + 1 < x$



b) . $(0,0)$ is not a/an interior / accumulation point

. $\text{int}\ \Omega = \{(x,y) \in \mathbb{R}^2 \ \ |x-5| < 1 \ \wedge \ 4 < y < 7\}$

. Ω is not open.

c) $\text{Im} f = [-4, 2]$

. $f \circ g$ has two zeros: e and e^{-1}

. $\lim\limits_{n \to +\infty} f\left(-\frac{1}{n}\right) = 2$

d) $f(x,y) = x^2 y^3 - \sin y$

e) $H_f(3,2) = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix}$ (for instance)

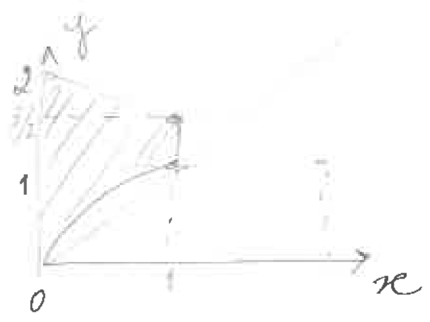
f) $\dfrac{\partial^{12} f}{\partial x^{10} \partial y^2}(x,y) = 3^8 \cos(3x)$

g) $M = \{ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : 1 \leq x^2 + y^2 \leq 2 \}$ (for instance)

Weierstrass

h) I_2 is not convex

i)



- $a \in [1, 3/2]$
- H_a is upper hemi-continuous
- $\mathrm{Fix}\, H_1 = \{0,1\}$

j) Banach fixed point theorem.

$$\lim_{n \to +\infty} f^n(0,0) = \underbrace{(2,2)}_{\text{fixed point of } f.}$$

k)
$$\begin{cases} 6y - 2 - 8\mu = 0 \\ 6x - 3\mu = 0 \\ \mu \leq 0 \\ \mu(8x + 3y - 17) = 0 \end{cases}$$

$$\boxed{\mu = 0}$$

l) separable

$$\frac{y'}{y^2} = -\frac{x}{x^2+1} \Leftrightarrow -\frac{1}{y} = -\frac{1}{2}\ln(x^2+1) + C$$

$C \in \mathbb{R}$

$$\Leftrightarrow y = \frac{1}{\frac{1}{2}\ln(x^2+1) + C}, \quad C \in \mathbb{R}$$

Since $y(0) = 2 \Leftrightarrow 2 = \dfrac{1}{\underbrace{\frac{1}{2}\ln 1}_{=0} + C} \Leftrightarrow C = \frac{1}{2}$

$$\boxed{y(x) = \frac{2}{\ln(x^2+1) + 1}}$$

$D = \mathbb{R}$

(m) $\begin{cases} y' = \underbrace{\sqrt{y-1}} \\ y(0) = 1 \end{cases}$

$f(y) = \sqrt{y-1}$, $D_f = [1, +\infty[$.

$f'(y) = \dfrac{1}{2\sqrt{y-1}}$

$\lim_{y \to 1^+} f'(y) = +\infty$

f is not Lipschitz at $y = 1$ ($x = 0$).

(n) (0,0) and (1,1) $\qquad f(x,y) = (-x+xy, y-xy)$

$$Df(x,y) = \begin{pmatrix} -1+y & x \\ -y & 1-x \end{pmatrix}$$

$$Df(1,1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \boxed{\begin{matrix} \dot{x} = y \\ \dot{y} = -x \end{matrix}}$$

(1,1) is non-hyperbolic because the eigenvalues of $Df(1,1)$ are $\pm i$ (they have zero real part).

(o) $p' = 20p - 10p^2 \iff p' = 10p(2-p)$.

equilibria $p=0$ and $p=2$

- monotonic increasing
- $\lim_{t\to+\infty} p(t) = 2$
- inflexion point when $p=1$

(p) Hamiltonian:

$$\boxed{H(t,x,u,p) = 1 - tx - u^2 + p.u}$$

optimality

$$\frac{\partial H}{\partial u} = 0 \iff \boxed{-2u + p = 0} \Longleftarrow u = p/2$$

hamiltonian equation

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \iff \begin{cases} \dot{x} = u \\ \dot{p} = +t \end{cases} \iff \boxed{\begin{cases} \dot{x} = p/2 \\ \dot{p} = t \end{cases}}$$

Transversality condition

$$\boxed{p(1) = 0} \iff \frac{t^2}{2} + C_1 = 0 \iff C_1 = -1/2$$

$$\iff \boxed{\frac{t^2}{2} - \frac{1}{2} = 0}$$

$p(t) = \frac{t^2}{2} - \frac{1}{2}$

$\dot{x}(t) = \frac{t^2}{4} - \frac{1}{4} \Rightarrow x(t) = \frac{t^3}{12} - \frac{1}{4}t + C_2$

$\overset{x(0)=1}{\Longrightarrow} C_2 = 1$

$\therefore x^*(t) = \frac{t^3}{12} - \frac{1}{4}t + 1 \qquad u^*(t) = \frac{t^2}{4} - \frac{1}{4}$

H is concave in (x,u)

$\nabla H(x,u) = (-t \, ; -2u + p)$

$\mathbb{H}_H(x,u) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ eigenvalues: ≤ 0

Part II

1 a) * $[0,1]^2$ is a compact and convex (it is a square with boundary).

* F is continuous (the components are polynomial).

$* \quad F([0,1]^2) \subset [0,1]^2$

Proof:

$\cdot \ (x,y) \in [0,1]^2 \Longrightarrow 0 \leq x+y \leq 2$

$$\Rightarrow 0 \leq \frac{(x+y)^2}{4} \leq 1$$

$$\Rightarrow \frac{(x+y)^2}{4} \in [0,1]. \checkmark$$

$\cdot \ (x,y) \in [0,1]^2 \Rightarrow x \in [0,1] \checkmark$

b) $F(x,y) = (x,y) \Leftrightarrow \begin{cases} \frac{(x+y)^2}{4} = x \\ x = y \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x^2 = x \\ \text{---} \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=1 \\ y=1 \end{cases}$$

Fixed points of F: $(0,0)$ and $(1,1)$

2)

Lagrangian:

$$\mathcal{L}(x,y,z,\lambda) = \frac{x}{2} + \frac{y}{2} - 2z - \lambda\left(\frac{x^2}{2} + \frac{y^2}{4} + z^2 - 1\right)$$

$$\partial\mathcal{L}(x,y,z,\lambda) = \bar{0} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{1}{2} - \lambda x = 0 \\ \frac{1}{2} - \lambda \frac{y}{2} = 0 \\ -2 - 2\lambda z = 0 \\ \frac{x^2}{2} + \frac{y^2}{4} + z^2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{\lambda} \\ z = -\frac{1}{\lambda} \\ \frac{\left(\frac{1}{2\lambda}\right)^2}{2} + \frac{\left(\frac{1}{\lambda}\right)^2}{4} + \left(-\frac{1}{\lambda}\right)^2 - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ \text{---} \\ \frac{1}{8\lambda^2} + \frac{2}{8\lambda^2} + \frac{8}{8\lambda^2} = 1 \end{cases} \Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ \text{---} \\ \frac{11}{8\lambda^2} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ \text{---} \\ 8\lambda^2 = 11 \end{cases} \Leftrightarrow \begin{cases} x = \frac{\pm 1}{\cancel{2}\cdot\frac{11}{\cancel{2}\sqrt{2}}} \\ y = \pm\frac{2\sqrt{2}}{11} \\ z = \mp\frac{2\sqrt{2}}{11} \\ \lambda = \frac{\pm 11}{2\sqrt{2}} \end{cases}$$

Candidates:

$$A = \left(\frac{\sqrt{2}}{11} ; \frac{2\sqrt{2}}{11} ; -\frac{2\sqrt{2}}{11} \right) \text{ and}$$

$$B = \left(-\frac{\sqrt{2}}{11} ; -\frac{2\sqrt{2}}{11} ; \frac{2\sqrt{2}}{11} \right)$$

$$f(A) = \frac{\sqrt{2}}{22} + \frac{\sqrt{2}}{11} + \frac{4\sqrt{2}}{11} = \frac{\sqrt{2}+2\sqrt{2}+8\sqrt{2}}{11}$$

$$= \sqrt{2}$$

$$f(B) < 0$$

$\therefore$ global maximum is $\sqrt{2}$.

3

$$\left.\begin{array}{l} y_p(x) = 2x - 1 \\ y_p'(x) = 2 \\ y_p''(x) = 0 \end{array}\right\} \longrightarrow$$

$$y'' + \alpha y' + 6y = 12x + 4$$

$$0 + \alpha \cdot 2 + 6 \cdot (2x - 1) = 12x + 4$$

$$\Leftrightarrow +2\alpha - 6 = 4$$

$$\Leftrightarrow 2\alpha = 10$$

$$\Leftrightarrow \boxed{\alpha = 5}.$$

$$y'' + 5y' + 6y = 12x + 4$$

$$P(\lambda) = \lambda^2 + 5\lambda + 6$$

$$P(\lambda) = 0 \Leftrightarrow \lambda = -2 \vee \lambda = -3$$

Solution

$$y(x) = C_1 e^{-2x} + C_2 e^{-3x} + 2x - 1, \ x \in \mathbb{R}$$

We know that

$$\begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 - 1 = -1 \\ -2C_1 e^{-2x} - 3C_2 e^{-3x} + 2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1 + C_2 = 0 \\ -2C_1 - 3C_2 = -1 \end{cases} \Leftrightarrow \begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases}$$

$$\therefore y(x) = -e^{-2x} + e^{-3x} + 2x - 1, \quad x \in \mathbb{R}$$

4)

$$\begin{cases} x' = x + 2y \\ y' = -2x + y \end{cases}$$

a)

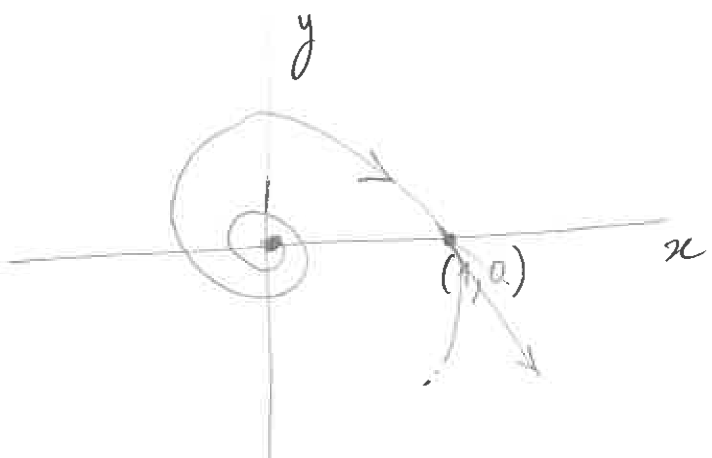
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 4$$

$$p(\lambda) = 0 \iff 1-\lambda = \pm 2i \iff \boxed{\lambda = 1 \pm 2i}$$

Since $Re(\lambda) = 1 > 0$, then $(0,0)$ is unstable.

b)



$x' = 1$
$y' = -2$

5

Euler-Lagrange equation: $F(t,x,\dot{x}) = 2tx + \dot{x}^2$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$$

$$\Leftrightarrow 2t - \frac{d}{dt}(2\dot{x}) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2t - 2\ddot{x} = 0 \Leftrightarrow \ddot{x} = t \Leftrightarrow \dot{x}(t) = \frac{t^2}{2} + C_1$$

$$\Leftrightarrow x(t) = \frac{t^3}{6} + C_1 t + C_2$$

Since:

$$\begin{cases} x(0) = 0 \\ x(1) = 7/6 \end{cases} \Leftrightarrow \begin{cases} C_2 = 0. \\ \frac{1}{6} + C_1 = \frac{7}{6} \end{cases} \Leftrightarrow \begin{cases} C_2 = 0. \\ C_1 = 1 \end{cases}$$

$$\therefore x(t) = \frac{t^3}{6} + t, \quad t \in [0,1]$$

$$\nabla F(x,\dot{x}) = (2t, 2\dot{x})$$

$$HF(x,\dot{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{eigenvalues } \geq 0$$

F is convex in $(x,\dot{x})$: $x^*(t) = \frac{t^3}{6} + t, \; t \in [0,1]$ is the solution.